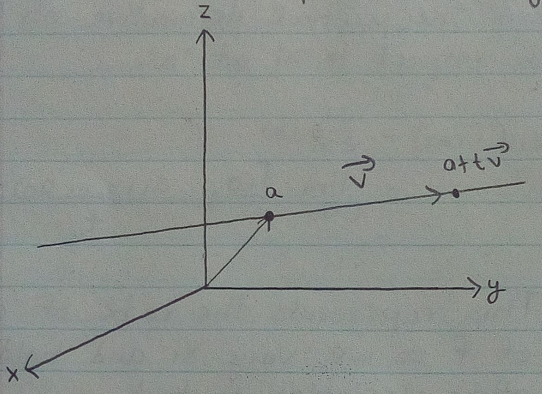
MATB41 Week 5 Notes

1. Differentiation:

Let $z = f(x)$. To find the rate of change in f at a point, a, along the line $a + t\vec{v}$, $t \in \mathbb{R}$:



$$\Delta x = (a + t\vec{v}) - a$$
$$= t\vec{v}$$

$$\Delta f = f(a + t\vec{v}) - f(a)$$

$$\frac{\Delta f}{\Delta x} = \lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t\|\vec{v}\|}$$

2. Directional Derivatives:

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. The directional derivative of f at "a" in direction $\vec{v}$, denoted by $D_v(f(a))$ is given by

$$D_v(f(a)) = \lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t\|\vec{v}\|}$$

Note: If v is a unit vector, then the formula is $\lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t}$.

Note: If $v = e_i$, $i = 1, 2, \ldots n$, $D_{e_i}(f(a))$ is denoted as $\frac{\partial f}{\partial x_i}(a)$ or $f_{x_i}(a)$ and is called the partial derivative of f with respect to x_i at "a".

$$D_{e_i}(f(a)) = \frac{\partial f}{\partial x_i}(a)$$
$$= \lim_{t \to 0} \frac{f(a_1, \ldots a_i + t, \ldots a_n) - f(a_1, \ldots a_n)}{t}$$

This represents the rate of change of f as we vary x_i and hold the other variables constant.

E.g. 1 Let $f(x,y,z) = x^2 - 2y + 3z^3$
Find the directional derivative of f at $(0,1,0)$ in the direction

a) $\vec{v} = [1,1,1]$

Solution:

$$\lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t\|\vec{v}\|}$$
$$= \lim_{t \to 0} \frac{f((0,1,0) + t[1,1,1]) - f(0,1,0)}{(t)(\sqrt{3})}$$
$$= \lim_{t \to 0} \frac{f(t, 1+t, t) - f(0,1,0)}{\sqrt{3}t}$$
$$= \lim_{t \to 0} \frac{t^2 - 2(1+t) + 3t^3 - (-2)}{\sqrt{3}t}$$
$$= \lim_{t \to 0} \frac{3t^3 - t^2 - 2t}{\sqrt{3}t}$$
$$= \lim_{t \to 0} \frac{3t^2 - t - 2}{\sqrt{3}} = -\frac{2}{\sqrt{3}}$$

b) $\vec{v} = [1,0,0]$

Since $\vec{v}$ is a unit vector, $\|\vec{v}\| = 1$.

Solution:

$$\lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t}$$

$$= \lim_{t \to 0} \frac{f((0,1,0) + t[1,0,0]) - f(0,1,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(t,1,0) - f(0,1,0)}{t}$$

$$= \lim_{t \to 0} \frac{t^2 - 2 - (-2)}{t}$$

$$= \lim_{t \to 0} \frac{t^2}{t}$$

$$= 0$$

E.g. 2 Calculate the partial derivatives of $f(x,y,z) = x^2 - 2y + 3z^3$.

Solution:

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = -2$$

$$\frac{\partial f}{\partial z} = 9z^2$$

3. Differentiability:

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. f is differentiable at $a \in U$ if

1. The partial derivatives of f exist at a.
2. $\lim_{x \to a} \frac{\| f(x) - f(a) - D(f(a))(x-a) \|}{\|x - a\|} = 0$

where $Df(a) = \left[\frac{\partial f_i}{\partial x_j} \right](a)$, $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$

is the Jacobian matrix of f at "a" given by

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} a & \frac{\partial f_1}{\partial x_2} a & \cdots & \frac{\partial f_1}{\partial x_n} a \\ \frac{\partial f_2}{\partial x_1} a & & & \vdots \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} a & & & \frac{\partial f_m}{\partial x_n} a \end{bmatrix}$$

or denoted as $Df(a) = \frac{\partial (f_1, f_2, \ldots, f_m)}{\partial (x_1, x_2, \ldots, x_n)} a$

E.g. 3 Calculate $Df(a)$ where $f(x,y,z) = (\underbrace{x^2 + y \sin z}_{f_1}, \underbrace{xe^y}_{f_2}, \underbrace{z \cos x}_{f_3})$ at $(1,1,1)$.

Solution:

$$Df(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

$$Df(a) = \begin{bmatrix} 2x & \sin z & y\cos z \\ e^y & xe^y & 0 \\ -z\sin x & 0 & \cos x \end{bmatrix}\Bigg|_{(1,1,1)}$$

$$= \begin{bmatrix} 2 & \sin(1) & \cos(1) \\ e & e & 0 \\ -\sin(1) & 0 & \cos(1) \end{bmatrix}$$

Note: When $m=1$, I.e. $f: U \subset \mathbb{R}^n \to \mathbb{R}$, $Df(a)$ is a $1 \times n$ matrix.

$Df(a) = \left[\frac{\partial f}{\partial x_1} a, \frac{\partial f}{\partial x_2} a, \ldots \frac{\partial f}{\partial x_n} a\right]$

This is the gradient of f at a, denoted by $\nabla f(a)$ or grad $f(a)$.

E.g. 4 Is the function $f(x,y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$ differentiable at $(0,0)$?

Solution:

We know that:

1. $f(x,0) = 0$
2. $f(0,y) = 0$
3. $f(0,0) = 0$

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x}$$

$$= 0$$

$$f_y(0,0) = \lim_{x \to 0} \frac{f(0,y) - f(0,0)}{y}$$

$$= 0$$

$$\therefore \frac{\partial f}{\partial x}\Big|_{(x,y)=(0,0)} = 0$$

$$\frac{\partial f}{\partial y}\Big|_{(x,y)=(0,0)} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\left\| x^{\frac{1}{3}}y^{\frac{1}{3}} - 0 - [0,0]\begin{bmatrix} x-0 \\ y-0 \end{bmatrix} \right\|}{\|(x,y)-(0,0)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\|x^{\frac{1}{3}}y^{\frac{1}{3}}\|}{\sqrt{x^2+y^2}}$$

The limit goes to infinity, so f is not differentiable at $(0,0)$.

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the partial derivatives of f all exist and are cont in the neighbourhood $a \in U$. Then, f is differentiable at $a \in U$.

If f is differentiable at "a", then:

1. All the partial derivatives of f at "a" exists.
2. f is cont at "a".

4. Properties of Derivatives:

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in U$. Let c be a constant. Then:

1. cf is diff at a and $D(cf)(a) = c(Df(a))$
2. $f \pm g$ is diff at a and $D(f\pm g)(a) = Df(a) \pm Dg(a)$
3. fg is diff at a and $D(fg)(a) = Df(a)g(a) + Dg(a)f(a)$
4. $\frac{f}{g}$ is diff at a if $g(a) \neq 0$ and

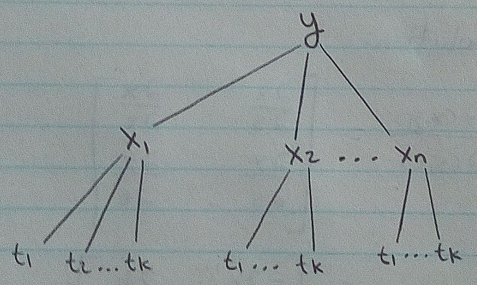
$$D\left(\frac{f}{g}\right)(a) = \frac{(Df(a))g(a) - f(a)(Dg(a))}{(g(a))^2}$$

The Chain Rule: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $g: V \subset \mathbb{R}^m \to \mathbb{R}^k$ be given functions s.t. f maps U to V so that $g \circ f$ is defined. Let $a \in U$ and $b \in V$. If f is diff at a and g is diff at b, then $f \circ g$ is diff at a and $D(g \circ f)(a) = (Dg(b))(Df(a))$.

Another way of thinking about the chain rule is:

Suppose that "y" is a diff function of n variables, $x_1, x_2, \ldots x_n$ and each x_i, $i = 1, \ldots, n$, is a diff function of k variables, $t_1, t_2, \ldots t_k$. Then, y is a function of the variables $t_1, t_2, \ldots t_k$ and

$$\frac{\partial y}{\partial t_j} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$



E.g. 5 Let $x = t^2 - s^2$, $y = ts$, $u = \sin(x+y)$, $v = \cos(x-y)$

a) Express (u,v) in terms of (s,t) and calculate $\frac{\partial(u,v)}{\partial(s,t)}$.

Solution:

$$u = \sin(t^2 - s^2 + st)$$
$$v = \cos(t^2 - s^2 - st)$$

$$\frac{\partial(u,v)}{\partial(s,t)} = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}$$

$$= \begin{bmatrix} (-2s+t)\cos(t^2-s^2+st) & (2t+s)\cos(t^2-s^2+st) \\ (-2s-t)(-\sin(t^2-s^2-st)) & (2t-s)(-\sin(t^2-s^2-st)) \end{bmatrix}$$

b) Compute $\frac{\partial(x,y)}{\partial(s,t)}$ and $\frac{\partial(u,v)}{\partial(x,y)}$

Solution:

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$= \begin{bmatrix} -2s & 2t \\ t & s \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & \cos(x+y) \\ -\sin(x-y) & \sin(x-y) \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(s,t)}$$

$$= \begin{bmatrix} \cos(x+y) & \cos(x+y) \\ -\sin(x-y) & \sin(x-y) \end{bmatrix} \begin{bmatrix} -2s & 2t \\ t & s \end{bmatrix}$$

$$= \begin{bmatrix} (-2s+t)(\cos(x+y)) & (2t+s)(\cos(x+y)) \\ (-2s-t)(-\sin(x-y)) & (2t-s)(-\sin(x-y)) \end{bmatrix}$$

$$= \frac{\partial(u,v)}{\partial(s,t)}$$